Professor F. Grillot

Problem set 3

EE 270 - Applied Quantum Mechanics

Due Wednesday Nov. 22, 2017 at 8.00 AM

Exercise I (20 points)

A function $|f\rangle$ can be expressed as an expansion of complete orthonormal basis functions $|\varphi_n\rangle$.

(a) Show that the identity operator $\hat{I} = \sum_n |\varphi_n\rangle\langle\varphi_n|$ acting on the function $|f\rangle$ leaves it unchanged.

(b) The sum of diagonal elements of an operator $\hat{A}$ expressed as a matrix is called a trace operator, $\mathrm{Tr}\left\{\hat{A}\right\}$. Show that the trace operator is independent of the basis used.

(c) A unitary operator satisfies $\hat{U}^{-1} = \hat{U}^{\dagger}$. Show that the inner product of functions $|f_1\rangle$ and $|g_1\rangle$ is invariant under unitary transformation such that $|f_2\rangle = \hat{U}|f_1\rangle$ and $|g_2\rangle = \hat{U}|g_1\rangle$.

(d) Demonstrate a unitary transformation can be used to change the representation of an operator from $\hat{A}$ to $\hat{B} = \hat{U}\hat{A}\hat{U}^{\dagger}$ by showing that the matrix elements satisfy $\langle g_1|\hat{A}|f_1\rangle = \langle g_2|\hat{B}|f_2\rangle$.

Exercise II (20 points)

Derive the following commutation relations.

$$[\hat{L}_z, \hat{x}] = i\hbar\hat{y}$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

$$[\hat{L}_z, \hat{z}] = 0$$

$$[\hat{L}_x, \hat{L}^2] = 0$$

From these relations, which physical quantities can be measured simultaneously ?

Exercise III (10 points)

The non-zero state $|n,t\rangle$ evolves in time according to the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|n,t\rangle = \hat{H}|n,t\rangle$ where $\hat{H}$ is the Hamiltonian. A unitary time-evolution operator seen in class $\hat{U}(t,t_0)$ evolves the state from time t_0 such that $|n,t\rangle = \hat{U}(t,t_0)|n,t_0\rangle$.

(a) Considering $\hat{H} \neq \hat{H}(t)$ show that $|n,t\rangle = \exp\Big(-i\hat{H}(t-t_0)/\hbar\Big)|n,t_0\rangle$.

(b) Considering $\hat{H} = \hat{H}(t)$ and $[\hat{H}(t),\hat{H}(t')] = 0$, and $t \neq t'$ show that $|n,t\rangle = \exp\left(\frac{-i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right)|n,t_0\rangle$.

Problem : Solving the mystery of the missing neutrinos (50 points)

Electron neutrinos are produced in the Sun as a product of nuclear fusion. Solar neutrinos constitute by far the largest flux of neutrinos from natural sources observed on Earth, as compared with e.g. atmospheric neutrinos or the diffuse supernova neutrino background. Neutrino oscillation is a quantum mechanical phenomenon

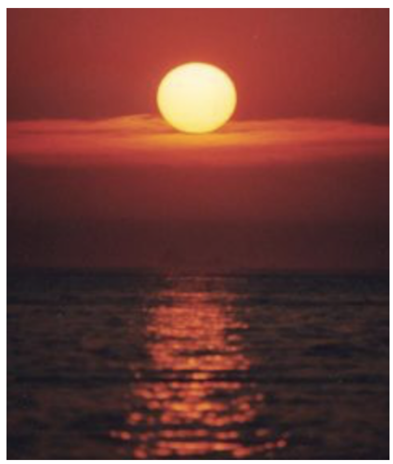

FIGURE 1 – Sunset in Santa Monica, California. Where are the solar missing neutrinos ?

whereby a neutrino created with a specific lepton flavor (electron, muon, or tau) can later be measured to have a different flavor. In 2015, the Nobel Prize in physics was awarded for the experimental observation of neutrino oscillations (see `https://physics.aps.org/articles/v8/97`) Neutrinos are ubiquitous chargeless elementary particles which can only be indirectly detected through their (rare) participation in nuclear decay processes. For many years, neutrinos were thought to be massless, but the discovery of neutrino oscillations confirmed that they do indeed have some (very small) finite mass. Although the experiments are complex, the basic theory of neutrino oscillations is relatively simple and can be understood using basic quantum mechanical concepts. For our purposes we need only consider two types of neutrinos, the electron and muon neutrinos ν_e and ν_μ

respectively, which are two orthonormal quantum states that comprise the basis of a 2-D Hilbert space.

Let us work with the basis states $|\nu_e\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $\left|\nu_\mu\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. The Hamiltonian in this basis for a neutrino at rest takes the form

$$H = c^2 \begin{pmatrix} m_e & m_x \\ m_x & m_\mu \end{pmatrix} \tag{1}$$

where c is the speed of light and m_e, m_μ, and m_x are real parameters.

(1) Find the energy eigenvalues E_1 and E_2 of the Hamiltonian (1). (Hint : recall that the eigenvalues λ of a matrix M can be found using the determinent equation $\det|M - \lambda I| = 0$.) If the mass of an energy eigenstate is given by $E_{1,2} = m_{1,2}c^2$, what are the masses m_1 and m_2 of the two energy eigenstates ?

(2) Find the corresponding energy eigenfunctions $|\nu_1\rangle$ and $|\nu_2\rangle$. Don't forget to normalize ! Write your answer in terms of a "mixing angle" θ defined such that

$$\cos\left(\frac{\theta}{2}\right) = \frac{m_x}{\sqrt{m_x^2 + (m_1 - m_e)^2}}$$
$$\sin\left(\frac{\theta}{2}\right) = \frac{m_1 - m_e}{\sqrt{m_x^2 + (m_1 - m_e)^2}}.$$

(Hint : if you first find $|\nu_1\rangle$, you can determine $|\nu_2\rangle$ with minimal calculation simply by requiring the two states to be orthogonal.)

(3) Write $\left|\nu_\mu\right\rangle$ and $|\nu_e\rangle$ in terms of the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$. Are $|\nu_e\rangle$ and $\left|\nu_\mu\right\rangle$ stationary states ?

(4) Suppose an electron neutrino is produced at time $t = 0$ in state $|\psi(t = 0)\rangle = |\nu_e\rangle$. What is the probability at a later time t that the state is observed as a muon neutrino, i.e., what is $|\left\langle \nu_\mu \middle| \psi(t) \right\rangle|^2$? The behavior described by this equation is called neutrino oscillation ; why do you think it has that name ? (Note that in this calculation $\left\langle \nu_\mu \right|$ is not time-dependent, but $|\psi(t)\rangle$ is.)

(5) Neutrinos travel at relativistic speeds such that their momentum $p \gg m_\nu c$ where m_ν is the mass of the neutrino. Then the total energy of each stationary state $E_{1,2} = pc\sqrt{1 + \frac{m_{1,2}^2 c^2}{p^2}} \approx pc + \frac{m_{1,2}^2 c^3}{2p}$. Using the result from part d), show that the probability of neutrino oscillation is proportional to

$$\sin^2\left(\frac{(m_2^2 - m_1^2)c^3 t}{4\hbar p}\right).$$

Comments

The result you have derived in this homework shows that 1) different types of neutrinos can transform into each other because the Hamiltonian "mixes" different types through the m_x term and 2) the rate of transformation is dependent on the differences in mass, implying that neutrinos are massive, so observation of neutrino oscillations is conclusive evidence of their mass. Notice that oscillations do not reveal the absolute values of m_1 and m_2; as far as I know, the latter are still experimentally undetermined (beyond upper bounds which indicate that they must be extremely small).

Two important sources of neutrinos that pass through Earth are μ_e generated by the Sun and particles generated in the upper atmosphere by cosmic rays, which produce a 2 :1 ratio of $\nu_\mu : \nu_e$. The Nobel Prize winning-experiments essentially measured the ratios of atmospheric (1998 at Kamiokande) and solar (2001 at Sudbury) ν_μ and ν_e that reach the surface of the Earth, finding them in each case to be different from the generation ratios and consistent with neutrino oscillations [1].

Further readings

(a) The Kamland detector in Japan is a an electron antineutrino detector at the Kamioka Observatory, an underground neutrino detection facility near Toyama, Japan.

`http://www.awa.tohoku.ac.jp/kamland/`

(b) Super-Kamiande is a neutrino observatory located under Mount Ikeno in Japan. The observatory was designed to detect high-energy neutrinos to search for proton decay, study solar and atmospheric neutrinos, and keep watch for supernovae in the Milky Way Galaxy.

`http://www-sk.icrr.u-tokyo.ac.jp/sk/index-e.html`

1. It has been known for decades that the number of solar electron neutrinos observed on Earth is significantly smaller than what is expected to reach our planet from the Sun. We now know the reason for this discrepancy is because many of these electron neutrinos are converted into muon neutrinos in route via neutrino oscillations